

The spectrum of an I -graph

Allana S. S. de Oliveira¹ and Cybele T. M. Vinagre²

¹Instituto de Ciências Exatas, Universidade Federal Fluminense, Volta Redonda, Brasil. e-mail: allanasthel@id.uff.br

²Instituto de Matemática e Estatística, Universidade Federal Fluminense, Niterói, Brasil. e-mail: cybl@vm.uff.br

November 12, 2015

Abstract

We completely determine the spectrum of an I -graph, that is, the eigenvalues of its adjacency matrix. We apply our result to prove known characterizations of connectedness and bipartiteness in I -graphs by using an spectral approach. With our result, we also determine the nullity of a certain subfamily of I -graphs.

AMS Subject Classification: 05C50, 15B05.

Keywords: I -graph, generalized Petersen graph, nullity.

1 Introduction

The class of I -graphs was introduced in the Foster Census [4] as a natural generalization of the so called [18] generalized Petersen graphs and has attracted the attention of many graph theorists. Considerable study of I -graphs under algebraical, combinatorial and geometric approaches can be found in [2], [3], [11], [15] and references therein. In our work we investigate the I -graphs under an spectral approach.

In our work, we completely determine the eigenvalues of the adjacency matrix of an I -graph by using known properties of circulant and circulant block matrices. Furthermore, we apply our result and spectral graph techniques to give new proofs of known necessary and sufficient conditions for bipartiteness and connectedness of arbitrary I -graphs. Also, we establish the nullity, that is, the dimension of the eigenspace associated to the null eigenvalue, for a certain subfamily of I -graphs.

2 Preliminaries

The *adjacency matrix* $\mathbf{A}(G) = [a_{ij}]$ of an arbitrary simple graph G with vertices x_1, x_2, \dots, x_n , is the $n \times n$ matrix where $a_{ij} = 1$, if $\{x_i, x_j\}$ is an edge of G , and $a_{ij} = 0$ otherwise. Since a simple graph has no loops or undirected edges, its adjacency matrix is symmetric and has zero diagonal elements. The *characteristic polynomial* $p_G(\lambda)$ of G is that of $\mathbf{A}(G)$, that is, $p_G(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}(G))$. An *eigenvalue* of G is any root of its characteristic polynomial. Since $\mathbf{A}(G)$ is symmetric, all the eigenvalues of G are real numbers. The *spectrum* of G is the set of its eigenvalues together with their multiplicities.

A *walk* of length k from vertex x_0 to x_k , denoted by $x_0 \sim x_1 \sim x_2 \sim \dots \sim x_{k-1} \sim x_k$, is a sequence of vertices $x_0, x_1, x_2, \dots, x_{k-1}, x_k$ such that the subsets $\{x_0, x_1\}, \{x_1, x_2\}, \dots, \{x_{k-1}, x_k\}$ are all edges. A *path* is a walk with all vertices (and hence all edges) distinct. A *cycle* is a path from a vertex back to itself (so the first and last vertices coincide). The cycle on n vertices is denoted C_n .

2.1 Circulant matrices

We may recall known facts about circulant and block circulant matrices. For more details, see for example [7], [13] and [17].

A square matrix in which each row (after the first) has the elements of the previous row shifted cyclically one place right is called a *circulant* matrix. So, a circulant matrix has the form

$$\mathbf{M} = \text{circ}(m_0, m_1, \dots, m_{n-1}) = \begin{bmatrix} m_0 & m_1 & \dots & m_{n-1} \\ m_{n-1} & m_0 & \dots & m_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & \dots & m_0 \end{bmatrix},$$

that is, $\mathbf{M} = [m_{i,j}] = [m_{0,j-i}]$, where the subscripts are reduced modulo n .

It is easy to see that $\mathbf{M} = \text{circ}(0, 1, 0, \dots, 0, 1)$ is the adjacency matrix of C_n . The following result can be found in [13]. For concreteness, we express the n th roots of unity as powers of $\xi = e^{\frac{2\pi i}{n}}$, where $i^2 = -1$.

Proposition 1 *The eigenvalues of $\mathbf{M} = \text{circ}(m_0, m_1, \dots, m_{n-1})$ are*

$$\lambda_l = m_0 + \xi^l m_1 + \dots + \xi^{(n-1)l} m_{n-1},$$

for $0 \leq l \leq n-1$, each one with corresponding eigenvector $\mathbf{v}_l = [1, \xi^l, \xi^{2l}, \dots, \xi^{(n-1)l}]^\top$. If $\mathbf{V} = [\mathbf{v}_0 \mid \mathbf{v}_1 \mid \dots \mid \mathbf{v}_{n-1}]$ is the matrix whose columns

are the vectors \mathbf{v}_l then $\mathbf{V}' = \frac{1}{\sqrt{n}}\mathbf{V}$ is a unitary matrix and $(\mathbf{V}')^{-1}\mathbf{M}\mathbf{V}'$ is a diagonal matrix, that is, \mathbf{V}' (unitarily) diagonalizes \mathbf{M} . For symmetric circulant matrices, the eigenvalues are real numbers.

2.2 I -graphs

Let fix $n, j, k \in \mathbb{N}$ with $n \geq 3$, $1 \leq j, k < \frac{n}{2}$ and $j \leq k$. The I -graph $I(n, j, k)$ is the graph with vertex set $V(I(n, j, k)) = \{a_i, b_i; 0 \leq i \leq n-1\}$ and edge set $E(I(n, j, k)) = \{\{a_i, a_{i+j}\}, \{a_i, b_i\}, \{b_i, b_{i+k}\}; 0 \leq i \leq n-1\}$, where addition is performed modulo n .

We may assume that $j \leq k$ since $I(n, j, k) = I(n, k, j)$. We consider $j, k < \frac{n}{2}$ because $I(n, j, k)$, $I(n, n-j, k)$ and $I(n, j, n-k)$ are isomorphic (neither $j = n/2$ nor $k = n/2$ furnish simple graphs). Thus, $I(n, j, k)$ are cubic graphs (that is, regular graphs of degree 3) on $2n$ vertices. The *Petersen graph* is $I(5, 1, 2)$. The class of I -graphs contains the well known class of $G(n, k) = I(n, 1, k)$, the so called ([18]) *generalized Petersen graphs*, introduced in [5]. For more on graphs $G(n, k)$ see also [1], [8], [9], [12], [14] and references therein.

The vertices $\{a_i; 0 \leq i \leq n-1\}$ and edges $\{\{a_i, a_{i+j}\}; 0 \leq i \leq n-1\}$ of the I -graph $I(n, j, k)$ form a subgraph which we denote by $A(n, j)$. On the other hand, the subgraph of $I(n, j, k)$ with vertices $\{b_i; 0 \leq i \leq n-1\}$ and edges $\{\{b_i, b_{i+k}\}; 0 \leq i \leq n-1\}$ will be denoted $B(n, k)$. The following proposition gives us an idea of the structures of these two subgraphs respecting connectivity.

Proposition 2 *The subgraph $B(n, k)$ has $d_B = \gcd(n, k)$ connected components, each one isomorphic to a cycle of length $\frac{n}{d_B}$. Analogously, the subgraph $A(n, j)$ has $d_A = \gcd(n, j)$ connected components, each one isomorphic to a cycle of length $\frac{n}{d_A}$.*

Proof: Let $q = \frac{n}{d_B} \in \mathbb{N}$. For $k = 1$, $\gcd(n, k) = 1$ and then, by construction, the graph $B(n, k)$ has a connected component, namely, the cycle C_n . For $k > 1$, the vertex b_0 is adjacent to b_{0+k} , which in turn is adjacent to b_{2k} , and so on, until vertex $b_{(q-1)k}$, which is adjacent to b_{qk} . But b_{qk} coincides with b_0 , since $qk \equiv 0 \pmod{n}$: indeed, $qk = n \frac{k}{d_B}$ with $\frac{k}{d_B} \in \mathbb{N}$. Then two cases are possible, depending on $\gcd(n, k) = 1$ or $\gcd(n, k) > 1$. If $\gcd(n, k) = 1$, then $qk = n \cdot k$ and $q = n$; since we go through all the vertices in $B(n, k)$, we have a connected component which is a cycle of length q . In case $d_B = \gcd(n, k) > 1$, it follows that $q = \frac{n}{d_B} < n$ and then, the closed path on q vertices from b_0 to $b_{qk} = b_0$ is covered. Thus we obtain a cycle

isomorphic to C_q . Starting again from the first vertex to the right of b_0 not in the cycle $b_0 \dots b_{qk}$, say b_l , we obtain $b_l \sim b_{l+k} \sim b_{l+2k} \sim \dots \sim b_{l+qk} = b_l$, a cycle of length q (like before). Since we have $n = qd_B$ vertices, we may proceed until we obtain d_B cycles of length q .

Analogously, it can be shown that the subgraph $A(n, j)$ has $d_A = \gcd(n, j)$ connected components, each one isomorphic to a cycle of length $\frac{n}{d_A}$.

□

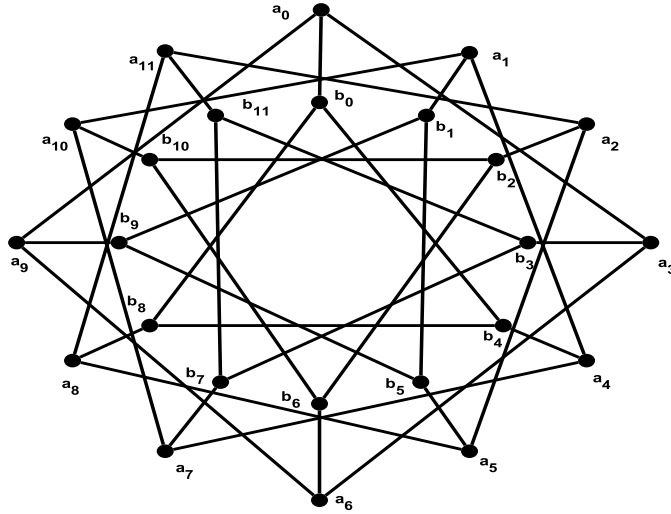


Figure 1: $I(12, 3, 4)$

Example 1 Figure 1 depicts the I -graph $I(12, 3, 4)$ and the vertices a_i and b_i of subgraphs $A(12, 3)$ and $B(12, 4)$, respectively.

It follows from the above proposition (or, alternatively, from the I -graph construction) that the adjacency matrices of the subgraphs $A(n, j)$ and $B(n, k)$ are circulant matrices. Indeed,

$$\mathbf{A}(A(n, j)) = \text{circ}(\overbrace{0, \dots, 0}^{j \text{ entries}}, 1, 0, \dots, 0, 1, \overbrace{0, \dots, 0}^{j-1 \text{ entries}}) \quad \text{and}$$

$$\mathbf{A}(B(n, k)) = \text{circ}(\overbrace{0, \dots, 0}^{k \text{ entries}}, 1, 0, \dots, 0, 1, \overbrace{0, \dots, 0}^{k-1 \text{ entries}}).$$

For instance, for $I(12, 3, 4)$ it holds that $\mathbf{A}(A(12, 3)) = \text{circ}(0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0)$ and $\mathbf{A}(B(12, 4)) = \text{circ}(0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0)$.

For brevity, we denote $\mathbf{A}^{nj} = \mathbf{A}(A(n, j))$ and $\mathbf{B}^{nk} = \mathbf{A}(B(n, k))$. The eigenvalues and eigenvectors of these matrices may be determined by using Proposition 1.

Lemma 3 *The eigenvalues of \mathbf{A}^{nj} are $\alpha_l = 2 \cos(\frac{2\pi jl}{n})$, with corresponding eigenvectors $\mathbf{v}_l = (1, \xi^l, \xi^{2l}, \dots, \xi^{(n-1)l})^\top$, $0 \leq l \leq n-1$. Analogously, \mathbf{B}^{nk} has eigenvalues $\beta_l = 2 \cos(\frac{2\pi kl}{n})$, with corresponding eigenvectors $\mathbf{v}_l = (1, \xi^l, \xi^{2l}, \dots, \xi^{(n-1)l})^\top$, $0 \leq l \leq n-1$.*

Proof: By Proposition 1, the eigenvalues of \mathbf{A}^{nj} are $\alpha_l = \xi^{jl} + \xi^{(n-j)l}$, $0 \leq l \leq n-1$, where

$$\begin{aligned} \alpha_l &= \xi^{jl} + \xi^{(n-j)l} \\ &= \cos\left(\frac{2\pi jl}{n}\right) + i \sin\left(\frac{2\pi jl}{n}\right) + \cos\left(\frac{2\pi l j(n-j)}{n}\right) + i \sin\left(\frac{2\pi l j(n-j)}{n}\right) \\ &= \cos\left(\frac{2\pi jl}{n}\right) + i \sin\left(\frac{2\pi jl}{n}\right) + \cos\left(\frac{2\pi jl}{n}\right) - i \sin\left(\frac{2\pi jl}{n}\right) \\ &= 2 \cos\left(\frac{2\pi jl}{n}\right). \end{aligned}$$

The other assertion follows similarly. □

3 Main result

Let consider the I -graph $I(n, j, k)$ fixed, where $n, j, k \in \mathbb{N}$ are as before agreed. We label the vertices of $I(n, j, k)$ as follows: the vertices b_0, \dots, b_{n-1} of $B(n, k)$ are designated, respectively, as $0, 1, 2, \dots, s, \dots, n-1$, and the vertices a_0, \dots, a_{n-1} in $A(n, k)$ are labelled, respectively, as $n, n+1, n+2, \dots, n+s, \dots, 2n+1$. From the adjacency rules in subgraphs $B(n, k)$ and $A(n, j)$ of $I(n, j, k)$, it follows respectively that $i \sim (i+k) \bmod n$ and $(n+i) \sim (n+i+j) \bmod n$, for $0 \leq i \leq n-1$. Besides this, the adjacency rules between vertices of $A(n, j)$ and $B(n, k)$ are given by $i \sim (n+i)$, $0 \leq i \leq n-1$. This way, the adjacency matrix of $I(n, j, k)$ can be described as the following block circulant matrix

$$\mathbf{A}(I(n, j, k)) = \begin{pmatrix} \mathbf{B}^{nk} & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{A}^{nj} \end{pmatrix},$$

where \mathbf{I}_n is the identity matrix of order n .

Theorem 4 *The eigenvalues of $I(n, j, k)$ have the form*

$$\lambda_l = \cos\left(\frac{2\pi jl}{n}\right) + \cos\left(\frac{2\pi kl}{n}\right) \pm \sqrt{\left(\cos\left(\frac{2\pi jl}{n}\right) - \cos\left(\frac{2\pi kl}{n}\right)\right)^2 + 1},$$

for $0 \leq l \leq n-1$.

Proof: For a fixed l , $0 \leq l \leq n-1$ let

$$a_l = \frac{\beta_l - \alpha_l \pm \sqrt{(\alpha_l - \beta_l)^2 + 4}}{2} \text{ and } \lambda_l = \frac{\beta_l + \alpha_l \pm \sqrt{(\alpha_l + \beta_l)^2 - 4(\alpha_l^2 \beta_l^2 - 1)}}{2},$$

where α_l and β_l are as given in Lemma 3. We will show that λ_l is an eigenvalue of $I(n, j, k)$ with associated eigenvector $\mathbf{w}_l = \begin{pmatrix} a_l \mathbf{v}_l \\ \mathbf{v}_l \end{pmatrix}$, where $\mathbf{v}_l = (1, \xi^l, \xi^{2l}, \dots, \xi^{(n-1)l})^\top$. In fact, from Lemma 3, it follows that

$$\begin{aligned} \begin{pmatrix} \mathbf{B}^{nk} & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{A}^{nj} \end{pmatrix} \begin{pmatrix} a_l \mathbf{v}_l \\ \mathbf{v}_l \end{pmatrix} &= \begin{pmatrix} a_l \mathbf{B}^{nk} \mathbf{v}_l + \mathbf{v}_l \\ a_l \mathbf{v}_l + \mathbf{A}^{nj} \mathbf{v}_l \end{pmatrix} = \begin{pmatrix} a_l \beta_l \mathbf{v}_l + \mathbf{v}_l \\ a_l \mathbf{v}_l + \alpha_l \mathbf{v}_l \end{pmatrix} = \\ &= \begin{pmatrix} (a_l \beta_l + 1) \mathbf{v}_l \\ (a_l + \alpha_l) \mathbf{v}_l \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta_l - \alpha_l \pm \sqrt{(\alpha_l - \beta_l)^2 + 4}}{2} \cdot \beta_l + 1 \right) \mathbf{v}_l \\ \left(\frac{\beta_l - \alpha_l \pm \sqrt{(\alpha_l - \beta_l)^2 + 4}}{2} + \alpha_l \right) \mathbf{v}_l \end{pmatrix} = \\ &= \begin{pmatrix} \left(\frac{\beta_l^2 - \beta_l \alpha_l \pm \beta_l \sqrt{(\alpha_l - \beta_l)^2 + 4}}{2} + 1 \right) \mathbf{v}_l \\ \left(\frac{\beta_l + \alpha_l \pm \sqrt{(\alpha_l - \beta_l)^2 + 4}}{2} \right) \mathbf{v}_l \end{pmatrix} = \begin{pmatrix} \lambda_l a_l \mathbf{v}_l \\ \lambda_l \mathbf{v}_l \end{pmatrix} = \lambda_l \begin{pmatrix} a_l \mathbf{v}_l \\ \mathbf{v}_l \end{pmatrix}. \end{aligned}$$

Then λ_l is an eigenvalue of $I(n, j, k)$. Furthermore,

$$\begin{aligned} \lambda_l &= \frac{\beta_l + \alpha_l \pm \sqrt{(\alpha_l + \beta_l)^2 - 4(\alpha_l \beta_l - 1)}}{2} = \frac{\beta_l + \alpha_l \pm \sqrt{(\alpha_l - \beta_l)^2 + 4}}{2} = \\ &= \cos\left(\frac{2\pi jl}{n}\right) + \cos\left(\frac{2\pi kl}{n}\right) \pm \sqrt{\left(\cos\left(\frac{2\pi jl}{n}\right) - \cos\left(\frac{2\pi kl}{n}\right)\right)^2 + 1}. \end{aligned}$$

The assertion is proved.

□

In the following remark we show how to obtain λ_l and a_l for all l , $0 \leq l \leq n-1$.

Remark 5 *In order to find the numbers a_l and λ_l and the vectors \mathbf{w}_l in Theorem 4 we proceeded as follows: For a fixed l , $0 \leq l \leq n-1$ consider $\mathbf{w}_l = \begin{pmatrix} a_l \mathbf{v}_l \\ \mathbf{v}_l \end{pmatrix}$, where $\mathbf{v}_l = (1, \xi^l, \xi^{2l}, \dots, \xi^{(n-1)l})^\top$ and a_l is a constant to be determined. We are searching for the solutions of the matrix equation*

$$\mathbf{A}(I(n, j, k))\mathbf{w}_l = \lambda \mathbf{w}_l,$$

which is

$$\begin{pmatrix} \mathbf{B}^{nk} & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{A}^{nj} \end{pmatrix} \cdot \begin{pmatrix} a_l \mathbf{v}_l \\ \mathbf{v}_l \end{pmatrix} = \lambda \begin{pmatrix} a_l \mathbf{v}_l \\ \mathbf{v}_l \end{pmatrix}.$$

It suffices to solve the system

$$\begin{cases} a_l \mathbf{B}^{nk} \mathbf{v}_l + \mathbf{v}_l = \lambda a_l \mathbf{v}_l \\ a_l \mathbf{v}_l + \mathbf{A}^{nj} \mathbf{v}_l = \lambda \mathbf{v}_l \end{cases}.$$

From Lemma 3 we have:

$$\begin{cases} a_l \beta_l \mathbf{v}_l + \mathbf{v}_l = \lambda a_l \mathbf{v}_l \\ a_l \mathbf{v}_l + \alpha_l \mathbf{v}_l = \lambda \mathbf{v}_l \end{cases} \Leftrightarrow \begin{cases} a_l(\lambda - \beta_l) \mathbf{v}_l = \mathbf{v}_l \\ (\lambda - \alpha_l) \mathbf{v}_l = a_l \mathbf{v}_l \end{cases} \Leftrightarrow \begin{cases} a_l(\lambda - \beta_l) = 1 \\ (\lambda - \alpha_l) = a_l \end{cases}$$

By substituting the value a_l into the first equation, we obtain

$$\begin{cases} (\lambda - \beta_l)(\lambda - \alpha_l) = 1 & (*) \\ (\lambda - \alpha_l) = a_l & (**) \end{cases}$$

Equation (*) furnishes

$$\lambda_l^+ = \frac{\beta_l + \alpha_l + \sqrt{(\alpha_l + \beta_l)^2 - 4(\alpha_l \beta_l - 1)}}{2} \quad \text{and} \\ \lambda_l^- = \frac{\beta_l + \alpha_l - \sqrt{(\alpha_l + \beta_l)^2 - 4(\alpha_l \beta_l - 1)}}{2}.$$

Then, from equation (**) we have the following values for a_l :

$$a_l^+ = (\lambda_l^+ - \alpha_l) = \frac{\beta_l - \alpha_l + \sqrt{(\alpha_l - \beta_l)^2 + 4}}{2} \quad \text{and} \\ a_l^- = (\lambda_l^- - \alpha_l) = \frac{\beta_l - \alpha_l - \sqrt{(\alpha_l - \beta_l)^2 + 4}}{2}$$

Thus,

$$\begin{aligned}\lambda_l^+ &= \frac{\beta_l + \alpha_l + \sqrt{(\alpha_l + \beta_l)^2 - 4(\alpha_l^2 \beta_l^2 - 1)}}{2} = \frac{\beta_l + \alpha_l + \sqrt{(\alpha_l - \beta_l)^2 + 4}}{2} = \\ &= \cos\left(\frac{2\pi jl}{n}\right) + \cos\left(\frac{2\pi kl}{n}\right) + \sqrt{\left(\cos\left(\frac{2\pi jl}{n}\right) - \cos\left(\frac{2\pi kl}{n}\right)\right)^2 + 1},\end{aligned}$$

Then λ_l^+ is an eigenvalue of $\mathbf{A}(I(n, j, k))$ with associated eigenvector $\mathbf{w}_l^+ = \begin{pmatrix} a_l^+ \mathbf{v}_l \\ \mathbf{v}_l \end{pmatrix}$. Similarly,

$$\begin{aligned}\lambda_l^- &= \frac{\beta_l + \alpha_l - \sqrt{(\alpha_l + \beta_l)^2 - 4(\alpha_l^2 \beta_l^2 - 1)}}{2} \\ &= \cos\left(\frac{2\pi jl}{n}\right) + \cos\left(\frac{2\pi kl}{n}\right) - \sqrt{\left(\cos\left(\frac{2\pi jl}{n}\right) - \cos\left(\frac{2\pi kl}{n}\right)\right)^2 + 1},\end{aligned}$$

that is, λ_l^- is an eigenvalue of $\mathbf{A}(I(n, j, k))$ with corresponding eigenvector $\mathbf{w}_l^- = \begin{pmatrix} a_l^- \mathbf{v}_l \\ \mathbf{v}_l \end{pmatrix}$. Then, for all l , $0 \leq l \leq n-1$, λ_l^+, λ_l^- are the $2n$ eigenvalues of $I(n, j, k)$.

Corollary 6 *The $2n$ eigenvalues of $I(n, j, k)$ are the roots of the equations $(\lambda - \beta_l)(\lambda - \alpha_l) = 1$, where $\alpha_l = 2\cos(\frac{2\pi jl}{n})$ and $\beta_l = 2\cos(\frac{2\pi kl}{n})$, for all l , $0 \leq l \leq n-1$.*

Proof: It follows from equation (*). □

The spectrum of a $G(n, k)$ graph was determined in [8], where the case $\gcd(n, k) = 1$ was completely proved and the other case had its proof outlined. Since $G(n, k) = I(n, 1, k)$, we have:

Corollary 7 ([8]) *The eigenvalues of $G(n, k)$ are, for all $0 \leq l \leq n-1$,*

$$\lambda_l = \cos\left(\frac{2\pi l}{n}\right) + \cos\left(\frac{2\pi kl}{n}\right) \pm \sqrt{\left(\cos\left(\frac{2\pi l}{n}\right) - \cos\left(\frac{2\pi kl}{n}\right)\right)^2 + 1}.$$

4 Two structural properties of I -graphs through spectral approach

The following are known structural properties of I -graphs, which have been demonstrated in [2] using combinatorial and graph-theoretical ideas. After our Theorem 4, we can prove them by using an spectral approach.

To prove the next theorem we use the known fact ([6], Theorem 3.23) that a regular graph of degree r is connected if and only if its (largest) eigenvalue r has multiplicity equal to one.

Theorem 8 ([2]) $I(n, j, k)$ is a connected graph if and only if $\gcd(n, j, k) = 1$.

Proof: Consider the graph $I(n, j, k)$ and let $d = \gcd(n, j, k)$. Suppose $d > 1$. For each $l = q\frac{n}{d}$, where $q = 0, \dots, d-1$, it follows from Theorem 4 that

$$\begin{aligned}\lambda_l &= \cos(2\pi jq) + \cos(2\pi kq) + \sqrt{(\cos(2\pi jq) - \cos(2\pi kq))^2 + 1} \\ &= \cos(2q'\pi) + \cos(2q''\pi) + \sqrt{(\cos(2q'\pi) - \cos(2q''\pi))^2 + 1} \\ &= 3,\end{aligned}$$

where $q' = j/d$ and $q'' = k/d$. Thus 3 is an eigenvalue of multiplicity equal to d and therefore, $I(n, j, k)$ has d connected components. Since $d > 1$, $I(n, j, k)$ is disconnected.

Conversely, suppose $\gcd(n, j, k) = 1$. Then $\gcd(n, j) = 1$ or $\gcd(n, k) = 1$ or $\gcd(j, k) = 1$. We claim that, in any case, if $0 \leq l \leq n-1$ and $\lambda_l^+ = 3$ then $l = 0$ (we note that $\lambda_l^- \leq 1$). Indeed, $\lambda_l^+ = 3$ if and only if

$$3 = \cos\left(\frac{2\pi jl}{n}\right) + \cos\left(\frac{2\pi kl}{n}\right) + \sqrt{\left(\cos\left(\frac{2\pi jl}{n}\right) - \cos\left(\frac{2\pi kl}{n}\right)\right)^2 + 1} \quad (1)$$

Denoting $x = \cos\left(\frac{2\pi jl}{n}\right)$ and $y = \cos\left(\frac{2\pi kl}{n}\right)$, we may solve the equation $3 = x + y + \sqrt{(x - y)^2 + 1}$, that is, $2xy - 3x - 3y + 4 = 0$, for $-1 \leq x, y \leq 1$. We write $y = \frac{3x-4}{2x-3} = f(x)$, $x \neq \frac{3}{2}$. Since $f'(x) = -\frac{1}{(3-2x)^2} < 0$, the function f decreases. Also $f(-1) = \frac{7}{5}$ and $f(1) = 1$, and thus there is no $x \in (-1, 1)$ such that $-1 \leq f(x) \leq 1$. Therefore, the only solution of the equation is $x = 1$ and $y = 1$. Returning to (1), the previous analysis leads us to

$$\begin{cases} (\cos(\frac{2\pi jl}{n}) - \cos(\frac{2\pi kl}{n}))^2 = 0 \\ \cos(\frac{2\pi jl}{n}) = 1 \\ \cos(\frac{2\pi kl}{n}) = 1 \end{cases} \Leftrightarrow \begin{cases} \frac{2\pi jl}{n} = 2\pi r \\ \frac{2\pi kl}{n} = 2\pi r' \end{cases} \Leftrightarrow \begin{cases} jl = rn \\ kl = r'n \end{cases}$$

for $r, r' \in \mathbb{Z}$. Thus $0 \leq \frac{rn}{j} \leq n-1$ and $0 \leq \frac{r'n}{k} \leq n-1$. We note that r is not a multiple of j , otherwise $\frac{rn}{j} \geq n$. Analogously, r' is not a multiple of k .

Let suppose $\gcd(n, j) = 1$. From $n|jl$ it follows that $n|l$ and then, $l < n$ implies $l = 0$. Similarly, if $\gcd(n, k) = 1$ holds then $l = 0$. For the case $\gcd(j, k) = 1$, let suppose that $l \neq 0$. Therefore r and r' are different from zero and, from the last system above, we obtain $jr' = kr$. Then the integer j can be written as $j = \frac{kr}{r'}$ and, since k is not a multiple of j , it follows that $r'|k$, say, $k = tr'$ for an integer $t > 1$. From $jr' = tr'r$ we have $j = tr$, and thus k and j have a common divisor $t > 1$, a contradiction. Then $l = 0$. Therefore the multiplicity of the eigenvalue 3 is equal to one and the graph $I(n, j, k)$ is connected.

□

In the sequence, we use a known characterization of bipartite graphs, which can be found in [6] (Theorem 3.11).

Proposition 9 *A graph containing at least one edge is bipartite if and only if its spectrum is symmetric with respect to zero.*

Theorem 10 ([2]) *The connected I -graph $I(n, j, k)$ is bipartite if and only if n is even and j and k are odd.*

Proof: Suppose that the connected graph $I(n, j, k)$ is bipartite. As we have seen, $\lambda_0^+ = 3$ is an eigenvalue of $I(n, j, k)$ of multiplicity one. Since $I(n, j, k)$ is bipartite, exists l , $0 \leq l \leq n-1$ such $\lambda_l^- = -3$ (we may note that $\lambda_l^+ \geq -1$, for all $0 \leq l \leq n-1$). For this fixed l it holds that

$$\cos\left(\frac{2\pi jl}{n}\right) + \cos\left(\frac{2\pi kl}{n}\right) - \sqrt{\left(\cos\left(\frac{2\pi jl}{n}\right) - \cos\left(\frac{2\pi kl}{n}\right)\right)^2 + 1} = -3.$$

Reasoning as in the proof of Theorem 8, we conclude that the only possibility for the above equation in the interval $[-1, 1]$ is

$$\begin{cases} \cos\left(\frac{2\pi jl}{n}\right) = -1 \\ \cos\left(\frac{2\pi kl}{n}\right) = -1 \\ \left(\cos\left(\frac{2\pi jl}{n}\right) - \cos\left(\frac{2\pi kl}{n}\right)\right)^2 = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{2\pi jl}{n} = \pi + 2\pi r \\ \frac{2\pi kl}{n} = \pi + 2\pi r' \end{cases},$$

for $r, r' \in \mathbb{Z}$. Thus $\frac{2jl}{n} = 1 + 2r$ and $\frac{2kl}{n} = 1 + 2r'$. Therefore, $2jl = n(1 + 2r)$, and then n is even. Also we have $j(1 + 2r) = k(1 + 2r')$, and so j and k

have the same parity. But if both the integers are even, it follows that $d = \gcd(n, j, k) > 1$, a contradiction according to Theorem 8, since the graph is connected. It follows that n is even and j and k are odd.

For the converse, suppose that n is even and let $l \in \mathbb{N}$ be such that $0 \leq l \leq \frac{n}{2}$. From trigonometry, for odd integers j and k we have

$$\cos\left(\frac{2\pi j\left(\frac{n}{2}-l\right)}{n}\right) = \cos(\pi j) \cos\left(\frac{2\pi jl}{n}\right) - \sin(\pi j) \sin\left(\frac{2\pi jl}{n}\right) = -\cos\left(\frac{2\pi jl}{n}\right)$$

$$\text{and } \cos\left(\frac{2\pi k\left(\frac{n}{2}-l\right)}{n}\right) = -\cos\left(\frac{2\pi kl}{n}\right).$$

Thus, from Theorem 4, it follows that

$$\lambda_l^+ + \lambda_{\frac{n}{2}-l}^- = \cos\left(\frac{2\pi jl}{n}\right) + \cos\left(\frac{2\pi kl}{n}\right) + \sqrt{\left(\cos\left(\frac{2\pi jl}{n}\right) - \cos\left(\frac{2\pi kl}{n}\right)\right)^2 + 1}$$

$$- \cos\left(\frac{2\pi jl}{n}\right) - \cos\left(\frac{2\pi kl}{n}\right) - \sqrt{\left(-\cos\left(\frac{2\pi jl}{n}\right) + \cos\left(\frac{2\pi kl}{n}\right)\right)^2 + 1} = 0,$$

Also, if l satisfies $\frac{n}{2} + 1 \leq l \leq n - 1$ then $\lambda_l^+ + \lambda_{\frac{3n}{2}-l}^- = 0$, since we have

$$\cos\left(\frac{2\pi k\left(\frac{3n}{2}-l\right)}{n}\right) = -\cos\left(\frac{2\pi kl}{n}\right)$$

$$\text{and } \cos\left(\frac{2\pi j\left(\frac{3n}{2}-l\right)}{n}\right) = -\cos\left(\frac{2\pi jl}{n}\right).$$

Thus, the spectrum is symmetric about the origin and $I(n, j, k)$ is a bipartite graph.

□

We may note that the above result does not apply to disconnected graphs. For instance, the disconnected graph $I(12, 2, 2)$ is bipartite, since it is formed from two copies of $I(6, 1, 1)$, which is bipartite.

The *nullity* $\eta = \eta(G)$ of a graph G is the multiplicity of the number zero in its spectrum. This graph-spectrum based invariant has a noteworthy application in chemistry and also in mathematics as we can learn in the survey [10] and the references therein. In [16], the author investigate multiplicity of eigenvalues of arbitrary cubic graphs. Applied to an arbitrary I -graph, the main result of this paper concerning nullity asserts that

Proposition 11 ([16]) *The nullity η of a connected I -graph $I(n, j, k)$ satisfies $\eta \leq n + 1$.*

In the sequence, we determine the nullity of I -graphs of the form $I(n, j, 2j)$ by using our Theorem 4.

Lemma 12 *There exists l , $0 \leq l \leq n - 1$, such that $\lambda_l = 0$ is an eigenvalue of the I -graph $I(n, j, 2j)$ if and only if exists l , $0 \leq l \leq n - 1$, such that $3jl = n(1 + 3r)$ or $3jl = n(2 + 3r)$ or $10jl = n(1 + 10r)$ or $10jl = n(3 + 10r)$ or $10jl = n(7 + 10r)$ or $10jl = n(9 + 10r)$, where $r \in \mathbb{Z}^+$ in each case.*

Proof: From Theorem 4, for $0 \leq l \leq n - 1$, the eigenvalues of $I(n, j, 2j)$ are of the form

$$\lambda_l^\pm = \cos\left(\frac{2\pi jl}{n}\right) + \cos\left(\frac{4\pi jl}{n}\right) \pm \sqrt{\left(\cos\left(\frac{2\pi jl}{n}\right) - \cos\left(\frac{4\pi jl}{n}\right)\right)^2 + 1}.$$

After algebraic manipulations, we may give the above equations the form

$$\lambda_l^\pm = 2x^2 + x - 1 \pm \sqrt{(2x^2 - x - 1)^2 + 1},$$

where $x = x(l) = \cos\left(\frac{2\pi jl}{n}\right)$ and $0 \leq l \leq n - 1$. Thus, finding l , $0 \leq l \leq n - 1$ such that $\lambda_l^\pm = 0$ leads us to deal with the roots of the equation $8x^3 - 4x - 1 = 0$, which are $-\frac{1}{2}$, $\frac{1+\sqrt{5}}{4}$ and $\frac{1-\sqrt{5}}{4}$. It is straightforward to verify that $\lambda_l^+ = 0$ if and only $x(l) = -\frac{1}{2}$ or $x(l) = \frac{1-\sqrt{5}}{2}$ and $\lambda_l^- = 0$ if and only $x(l) = \frac{1+\sqrt{5}}{2}$.

Let us analyse each case separately. Firstly,

$$\begin{aligned} & \exists l \in \mathbb{Z}, 0 \leq l \leq n - 1, x(l) = \cos\left(\frac{2\pi jl}{n}\right) = -\frac{1}{2} \\ \Leftrightarrow & \exists l \in \mathbb{Z}, 0 \leq l \leq n - 1, \frac{2\pi jl}{n} = \frac{2\pi}{3} + 2\pi r, r \in \mathbb{Z}^+, \vee \frac{2\pi jl}{n} = \frac{4\pi}{3} + 2\pi r, r \in \mathbb{Z}^+, \\ \Leftrightarrow & \exists l \in \mathbb{Z}, 0 \leq l \leq n - 1, 3jl = n(1 + 3r), r \in \mathbb{Z}^+, \vee 3jl = n(2 + 3r), r \in \mathbb{Z}^+. \end{aligned}$$

Reasoning analogously, we obtain the others two cases and complete the proof of our assertion:

$$\begin{aligned} & \exists l \in \mathbb{Z}, 0 \leq l \leq n - 1, \frac{2\pi jl}{n} = \frac{1-\sqrt{5}}{4} \\ \Leftrightarrow & \exists l \in \mathbb{Z}, 0 \leq l \leq n - 1, \frac{2\pi jl}{n} = \frac{3\pi}{5} + 2\pi r, r \in \mathbb{Z}^+, \vee \frac{2\pi jl}{n} = \frac{7\pi}{5} + 2\pi r, r \in \mathbb{Z}^+, \\ \Leftrightarrow & \exists l \in \mathbb{Z}, 0 \leq l \leq n - 1, 10jl = n(3 + 10r), r \in \mathbb{Z}^+, \vee 10jl = n(7 + 10r), r \in \mathbb{Z}^+, \end{aligned}$$

and

$$\begin{aligned} & \exists l \in \mathbb{Z}, 0 \leq l \leq n - 1, \frac{2\pi jl}{n} = \frac{1+\sqrt{5}}{4} \\ \Leftrightarrow & \exists l \in \mathbb{Z}, 0 \leq l \leq n - 1, \frac{2\pi jl}{n} = \frac{\pi}{5} + 2\pi r, r \in \mathbb{Z}^+, \vee \frac{2\pi jl}{n} = \frac{9\pi}{5} + 2\pi r, r \in \mathbb{Z}^+ \\ \Leftrightarrow & \exists l \in \mathbb{Z}, 0 \leq l \leq n - 1, 10jl = n(1 + 10r), r \in \mathbb{Z}^+, \vee 10jl = n(9 + 10r), r \in \mathbb{Z}^+. \end{aligned}$$

□

Theorem 13 *The nullity of the I-graph $I(n, j, 2j)$ is $\eta = |\mathcal{R}_1| + |\mathcal{R}_2| + |\mathcal{S}_1| + |\mathcal{S}_3| + |\mathcal{S}_7| + |\mathcal{S}_9|$, where $\mathcal{R}_i = \{r \in \mathbb{Z} : 3j|n(i+3r) \wedge 0 \leq r \leq j-1\}$ for $i = 1, 2$, $\mathcal{S}_t = \{s \in \mathbb{Z} : 10j|n(t+10s) \wedge 0 \leq s \leq j-1\}$, for $t = 1, 3, 7$, and $\mathcal{S}_9 = \{s \in \mathbb{Z} : 10j|n(9+10s) \wedge 0 \leq s \leq j-1\}$, if $n > 10j$, or $\mathcal{S}_9 = \{s \in \mathbb{Z} : 10j|n(9+10s) \wedge 0 \leq s \leq j-2\}$, if $n \leq 10j$.*

Proof: According Lemma 12, to calculate the nullity we must counting the number of indices l , $0 \leq l \leq n-1$, satisfying any of its conditions. We will try to rewrite each of them in a simpler way. We begin noting that from the above proof, provided l is an integer, the equality $3jl = n(1+3r)$ means that $3j|n(1+r)$ (then $l = \frac{n(1+r)}{3j}$) and also that the integer r satisfies $0 \leq r \leq \frac{3jn-3j-n}{3n} = j - \frac{j}{n} - \frac{1}{3}$, since $0 \leq l \leq n-1$. Furthermore, since $2j < n/2$ as agreed from the beginning of the article, then $j-1 < j - (\frac{1}{4} + \frac{1}{3}) < j - (\frac{j}{n} + \frac{1}{3}) < j$; so, it suffices to have the integer r satisfying $0 \leq r \leq j-1 = \lfloor j - \frac{j}{n} - \frac{1}{3} \rfloor$. More precisely, we just show that there exist $l, r \in \mathbb{Z}$ such that $0 \leq l \leq n-1$ and $3jl = n(1+3r)$ (and in this case $l = \frac{n(1+3r)}{3j}$) if and only if there exists $r \in \mathbb{Z}$ such that $3j|n(1+3r)$ and $0 \leq r \leq j-1$. This way, we obtain the elements of the set \mathcal{R}_1 . By proceeding analogously, we may determine the sets \mathcal{R}_2 , \mathcal{S}_1 , \mathcal{S}_3 and \mathcal{S}_7 and that $\mathcal{S}_9 = \{s \in \mathbb{Z} : 10j|n(9+10s) \wedge 0 \leq s \leq j - \lfloor j - \frac{j}{n} - \frac{9}{10} \rfloor\}$. Regarding the latter set, its description depends on having $n > 10j$ – when it suffices to take $0 \leq s \leq j-1$, – or $n \leq 10j$, when we may request $0 \leq s \leq j-2$. We note that in case all of the sets are empty, it follows from Lemma 12 that the I-graph $I(n, j, 2j)$ has nullity equal to zero.

□

Example 2 *For instance, $\eta(I(30, 2, 4)) = 4$, since we have*

$$\mathcal{R}_1 = \left\{ r \in \mathbb{Z} : 0 \leq r \leq 1 \wedge \frac{30(1+3r)}{6} \in \mathbb{Z} \right\} = \{0, 1\}$$

,

$$\mathcal{R}_2 = \left\{ r \in \mathbb{Z} ; 0 \leq r \leq 1 \wedge \frac{30(2+3r)}{6} \in \mathbb{Z} \right\} = \{0, 1\},$$

$$\mathcal{S}_1 = \left\{ r \in \mathbb{Z} ; 0 \leq r \leq 1 \wedge \frac{30(1+10r)}{20} \in \mathbb{Z} \right\} = \emptyset,$$

$$\mathcal{S}_3 = \left\{ r \in \mathbb{Z} ; 0 \leq r \leq 1 \wedge \frac{30(3+10r)}{20} \in \mathbb{Z} \right\} = \emptyset,$$

$$\mathcal{S}_7 = \left\{ r \in \mathbb{Z}; 0 \leq r \leq 1 \wedge \frac{30(7+10r)}{20} \in \mathbb{Z} \right\} = \emptyset$$

and

$$\mathcal{S}_9 = \left\{ r \in \mathbb{Z}; 0 \leq r \leq 1 \wedge \frac{30(9+10r)}{20} \in \mathbb{Z} \right\} = \emptyset.$$

The indices are $5^+, 20^+, 10^+$ and 25^+ (see the proof of Lemma 12). In turn, $\eta(I(30, 7, 14)) = 6$, since in this case,

$$\mathcal{R}_1 = \left\{ r \in \mathbb{Z} : 0 \leq r \leq 6 \wedge \frac{30(1+3r)}{21} \in \mathbb{Z} \right\} = \{2\},$$

$$\mathcal{R}_2 = \left\{ r \in \mathbb{Z}; 0 \leq r \leq 6 \wedge \frac{30(2+3r)}{21} \in \mathbb{Z} \right\} = \{4\},$$

$$\mathcal{S}_1 = \left\{ r \in \mathbb{Z}; 0 \leq r \leq 6 \wedge \frac{30(1+10r)}{70} \in \mathbb{Z} \right\} = \{4\},$$

$$\mathcal{S}_3 = \left\{ r \in \mathbb{Z}; 0 \leq r \leq 6 \wedge \frac{30(3+10r)}{70} \in \mathbb{Z} \right\} = \{6\},$$

$$\mathcal{S}_7 = \left\{ r \in \mathbb{Z}; 0 \leq r \leq 6 \wedge \frac{30(7+10r)}{70} \in \mathbb{Z} \right\} = \{0\}$$

and

$$\mathcal{S}_9 = \left\{ r \in \mathbb{Z}; 0 \leq r \leq 5 \wedge \frac{30(9+10r)}{70} \in \mathbb{Z} \right\} = \{4\}.$$

Here the indices are $10^+, 20^+, 9^+, 27^+, 3^-$ and 21^- .

5 Conclusion

In this paper, we completely determine the spectrum of an arbitrary I -graph and show how to use the theorem to obtain structural and spectral properties of these graphs.

References

- [1] Alspach, B., The classification of hamiltonian generalized Petersen graphs. *J. Combin. Theory, Ser. B*, v.34(3), 1983, 293-312.
- [2] Boben, M., Pisanski, T. and Žitnik, A., I -graphs and the corresponding configurations. *Journal of Combinatorial Designs* v.13, 2005, 406-424.

- [3] Bonvicini, S. and Pisanski, T., Hamiltonian cycles in I -graphs. *Electronic Notes in Discrete Mathematics*. v. 40, 2013, 43-47.
- [4] Bouwer, I.Z., Chernoff W.W., Monson, B. and Star, Z., The Foster Census: R.M. Foster's census of connected symmetric trivalent graphs. Charles Babbage Research Centre, 1988.
- [5] Coxeter, H.S.M. Self-dual configurations and regular graphs. *Bulletin of American Mathematical Society*, v. 56, 1950. 413-455.
- [6] D. Cvetković, M. Doob and H. Sachs, Spectra of Graphs - Theory and Application, Academic Press, New York, 1980.
- [7] Davis, P.R., Circulant Matrices, John Wiley, New York, 1979.
- [8] Gera, R. and Stănică, P., The spectrum of generalized Petersen graphs. *Australasian Journal of Combinatorics*. V. 49, 2011. p 39-45.
- [9] Frucht, R.; Graver, J. E.; and Watkins, M. E. The Groups of the Generalized Petersen Graphs. *Proc. Cambridge Philos. Soc.* 70, 1971. p 211-218.
- [10] Gutman, I. and Borovičanić, B.. Nullity of Graphs: An Updated Survey. in Selected Topics on Applications of Graph Spectra of Zbornik Radova series. v.14(22), 2011. 137-154.
- [11] Horvat, B, Pisanski, T. and Žitnik, A., Isomorphism checking of I -graphs, *Graphs and Combinatorics* (2012) 28, 823-830
- [12] Horvat, B, Pisanski, T. and Žitnik, A.. All generalized Petersen graphs are unit-distance graphs, *J. Korean Math. Soc.* V. 49, No. 3, (2012). p 475-491
- [13] Kra, I. and Simanca, S. R. On circulant matrices, *Notices of the American Mathematical Society*, V. 59(3), 2012. 368-377.
- [14] Mai, T.C., Wang, J.J. and Hsu, L.H., Hyper-Hamiltonian generalized Petersen graphs. *Computers and Mathematics with Applications* V. 55, 2008. 2076-2085.
- [15] Petkovšek, M., Zakrajšek, H.. Enumeration of I -graphs: Burnside does it again. *Ars Math. Contemp.* v.2, (2009) 241-262
- [16] Rowlinson, P., Eigenvalue multiplicity in cubic graphs. *Linear Algebra and Applications*. V.444, 2014, p 211-218.

- [17] Tee, G., Eigenvectors of block and alternating matrices, *Res. Lett. Inf. Math. Sci.*, Vol.8, 2005. 123-142.
- [18] Watkins, M. E., A Theorem on Tait Colorings with an Application to the Generalized Petersen Graphs. *Journal of Combinatorial Theory*. V. 6, 1969. p 152-164.